

UDC 517.9

ON „UNIMODAL-BIMODAL” BIFURCATION
IN A FAMILY PIECEWISE LINEAR MAPSПРО БІФУРКАЦІЮ ПЕРЕХОДУ
ВІД УНІМОДАЛЬНОГО ВІДОБРАЖЕННЯ
ДО БІМОДАЛЬНОГО В КУСКОВО-ЛІНІЙНИХ
ВІДОБРАЖЕННЯХ

Yu.L. Maistrenko, V.L. Maistrenko, S.I. Popovych

Inst. Math. Nat. Acad. Sci. Ukraine,
Ukraine, 252601, Kyiv 4, Tereshchenkivs'ka str., 3
e-mail: maistrenko@imath.kiev.ua;

vikul@imath.kiev.ua

We prove the theorem on existence of period adding cascades of attracting cycles $\gamma_{2+2k/5+4k}$ and $\gamma_{1+2k/3+4k}$, $k = 0, 1, \dots$, arising as a result of transition from unimodal to bimodal map in a family of three-piecewise linear one-dimensional maps.

Доводиться теорема про існування каскаду адитивного додавання періоду притягуючих циклів $\gamma_{2+2k/5+4k}$ та $\gamma_{1+2k/3+4k}$, $k = 0, 1, \dots$, що виникають внаслідок переходу від унімодального відображення до бімодального в сім'ї трьохкусових лінійних одновимірних відображень.

1. The problem. Consider a one-dimensional map arising from the time-delayed Chua's circuit [1], in the form:

$$f = f_{l,p,b}(x) = \begin{cases} f_1 := lx - lb - l/p + 1, & x \leq b + 1/p; \\ f_2 := px - pb, & b + 1/p \leq x \leq b - 1/p; \\ f_3 := lx - lb + l/p - 1, & x \geq b - 1/p, \end{cases} \quad (1)$$

where parameters l and p are slopes of linear branches and b is a shift with respect to the origin, of the graph $y = f(x)$. The parameters are supposed to belong the region

$$\Pi = \{(l, p, b) : 0 < l < 1, \quad -\infty < p < -1, \quad |b| < 1 + 1/p\}. \quad (2)$$

For all $(l, p, b) \in \Pi$, $f_{l,p,b}$ has a unique finite invariant interval $[-1, 1]$ (invariance means that $f([-1, 1]) = [-1, 1]$) where f is a bimodal three-piecewise linear map (Fig. 1, a). Interval $[-1, 1]$ attracts all trajectories of f ; more exactly, any trajectory gets into $[-1, 1]$ in a finite number of iterations and never leaves it after. An invariant set possessing such type property of attraction is, usually, called by *absorbing or trapping set* or area [2]. Thus, for any $(l, p, b) \in \Pi$ $f_{l,p,b}$ has an absorbing interval $[-1, 1]$ where it is three-piecewise one.

If

$$1 + 1/p < |b| < 1 - 1/p,$$

absorbing interval still exists, being equal $[f(1), 1]$ for $b > 0$ and $[-1, f(-1)]$ for $b < 0$, but the map considered is a two-piecewise there. Indeed, it is a skew tent map in the form

$$f_{l,p}(x) = \begin{cases} lx + a, & x \leq c; \\ px - p + f_{l,p,b}(1), & x \geq c, \end{cases} \quad (3)$$

where $a = 1 - l(b + 1/p)$, $c = b + 1/p$ (Fig. 1, b). Regular and chaotic attractors of the skew tent family were described in [3, 4].

At last, if

$$|b| > 1 - 1/p,$$

$f_{l,p,b}$ does not have any finite nontrivial invariant intervals, and the unique its attractor is an attractive fixed point x^* equals $\frac{lb + l/p - 1}{l - 1}$ for $b > 0$ or $\frac{lb - l/p + 1}{l - 1}$ for $b < 0$.

So, when parameter b crosses the surfaces $|b| = 1 - 1/p$ ($|b|$ decreases) „border-collision” bifurcation takes place [5]. It consists in passing by the attractive fixed point x^* through the extremum (turning) point which is a border of two intervals where the map is linear. Result of border-collision bifurcation is a birth of an absorbing invariant interval, and f is a skew tent map there. Just after the border-collision bifurcation, this newly appeared invariant interval is infinitesimal, then its size increases linearly with b .

Let's continue decreasing the modulus of the parameter b . When it passes through the value $1 + 1/p$, (l, p, b) -parameter point enters the region Π , and the „unimodal-bimodal” bifurcation takes place consisting in the following: in the invariant interval, the map $f_{l,p,b}$ turns to change its form from two-piecewise to three-piecewise ones.

2. Attracting cycles and their rotation numbers. At the parameter surface $p = -\infty$, the map $f = f_{l,p,b}$ can be represented as a discontinuous circle map in the form:

$$x \mapsto lx + a \pmod{1}, \quad (4)$$

where $a = 1 - l(b + 1)/2$. If $l = 1$, the map (4) turns to a shift circle map with a rotation number $\rho = (1 - b)/2$. At $l < 1$ regions $\Pi_{r/q}$ arise (Arnold's tongues) in the parameter plane (l, b) for any r and q be relatively prime. Each region $\Pi_{r/q}$ is characterized by existence of attracting cycle $\gamma_{r/q}$ with rotation number r/q .

Let now $p > -\infty$. Then $f_{l,p,b}$, $(l, p, b) \in \Pi$, has unique fixed point $x^* = pb/(p - 1)$ which is repelling. For any period- q cycle of the map

$$\gamma = \{f^n(x_0)\}_{n=0}^{n=q-1}, \quad (f^q(x_0) = x_0)$$

we introduce the following proper fraction:

$$\rho = \frac{\#\{i = \overline{1, q} : f^i(x) \in (x^*, 1]\}}{q} = \frac{r}{q} \quad (5)$$

to be called a *generalized rotation number* (or simply, *rotation number*) of the cycle γ . Denominator q of the fraction (5) equals to the period of the cycle. Numerator r indicates how many points of γ belong to the interval $(x^*, 1]$. Geometrically, r is a number of inverses of corresponding q -periodic trajectory of f . Note, that the fraction r/q may be reducible.

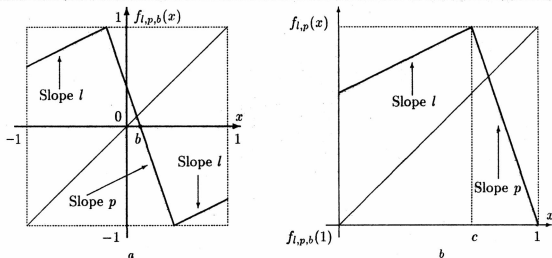


Fig. 1

At $p \rightarrow -\infty$ this generalized rotation number turns to usual rotation number of circle map [6], which is the same for any trajectory. For $f_{l,p,b}$ at $p > -\infty$ it is not the case: different trajectories may have different rotation numbers. Rotation numbers of all admissible trajectories of f forms *rotation interval* [6].

Let γ be a cycle with a rotation number $\rho = r/q$. Cycle, possessing this property, will be denoted by $\gamma_{r/q}$. Moreover, (l, p, b) -parameter region where $\gamma_{r/q}$ exists and is attracting will be denoted by $\Pi(\gamma_{r/q})$ or, simply $\Pi_{r/q}$.

As it was shown in [7], bifurcation „via $p = -\infty$ ” of the family $f_{l,p,b}$ gives rise to the regions $\Pi_{r/q}$. Every regions $\Pi_{r/q}$ is characterized by the existence of the only one attracting cycle $\gamma_{r/q}$ which is a continuation of the corresponding attracting cycle incoming from the case $p = -\infty$. The cycle $\gamma_{r/q}$ has r points belonging to the right segment $[b - 1/p; 1]$ and $q - r$ point belonging to the left segment $[-1; b + 1/p]$ and has not any points in the middle segment $[b + 1/p, b - 1/p]$ (for a definiteness, we assume $b > 0$).

3. „Unimodal-bimodal” bifurcation in the window between the regions $\Pi_{1/2}$ and $\Pi_{1/3}$. This bifurcation consists in the transition from unimodal i. e. skew tent map $f_{l,p}$ to bimodal map $f_{l,p,b}$ in the invariant interval of f . In parameter space (l, p, b) it occurs at $|b| = 1 + 1/p$.

Numerical evidence suggests that, generally, such type bifurcation results in period adding sequences of attracting point cycles $\gamma_{a_{11}+a_{12}k/a_{21}+a_{22}k}$, $k = 0, 1, \dots$, where a_{ij} , $i, j = 1, 2$ are integers characterizing each sequence [8–10]. All attracting cycles arising due to this bifurcation have the only one point belonging to the right segment $[b - 1/p, 1]$ ($b > 0$) or to the left segment $[-1, b + 1/p]$ ($b < 0$).

In the theorem below we describe two such sequences in the window between regions

$$\Pi_{1/2} = \left\{ \frac{l-1}{l+1}(1+1/p) < b < \frac{1-l}{1+l}(1+1/p) \right\}$$

and

$$\Pi_{1/3} = \left\{ \frac{1-l+l^2}{1+l+l^2} + \frac{(1-l-l^2)1/p}{1+l+l^2} < b < -\frac{1+l-l^2}{1+l+l^2} - \frac{(1-l-l^2)1/p}{1+l+l^2} \right\}$$

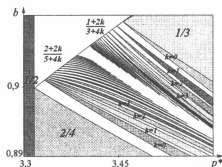


Fig. 2

(Fig. 2). Two sequences of regions of the existence of attracting cycles $\gamma_{\frac{2+2k_1}{3+4k_1}}$ and $\gamma_{\frac{1+2k_2}{3+4k_2}}$, $k_1, k_2 = 0, \dots$, are shown in the window between the regions $\Pi_{1/2}$ and $\Pi_{1/3}$ in the (p^*, b) parameter plane, where $p^* = \log_2(-p)$, at $l = 0, 1$.

Let $f_{l,p,b}, (l, p, b) \in \Pi$ be a map in the form (1). For any its trajectory $\eta = \{f^i(x)\}_{i=0}^{\infty}$, the following symbolic sequence

$$\xi_0 \xi_1 \dots \xi_n \dots$$

of five symbols L, M, R, C_1, C_2 can be constructed by the rule:

$$\xi_i = \begin{cases} L, & \text{if } f^i(x) \in [-1, b + 1/p]; \\ M, & \text{if } f^i(x) \in (b + 1/p, b - 1/p); \\ R, & \text{if } f^i(x) \in (b - 1/p, 1]; \\ C_1, & \text{if } f^i(x) = b + 1/p; \\ C_2, & \text{if } f^i(x) = b - 1/p, \end{cases} \quad (6)$$

$$i = 0, 1, \dots$$

Theorem. When the parameter point (l, p, b) enters the region Π through the surface $b = 1 + 1/p$, between the regions $\Pi_{1/2}$ and $\Pi_{1/3}$, two sequences of attracting cycles $\gamma_{\rho_{k_1}}$ and $\gamma_{\rho_{k_2}}$ appear with the following generalized rotation numbers and symbolic sequences:

$$\rho_{k_1} = \frac{2 + 2k_1}{5 + 4k_1}, \quad (LM)^{2k_1} LM^2 LR \dots, \quad k_1 = 0, 1, \dots,$$

$$\rho_{k_2} = \frac{1 + 2k_2}{3 + 4k_2}, \quad (LM)^{2k_2} MLR \dots, \quad k_2 = 0, 1, \dots$$

Moreover:

i) numbers of attracting cycles γ_{k_1} and γ_{k_2} are finite at any fixed $l \in (0, 1)$ and tends to infinity when l tends to 0;

ii) regions $\Pi_{\rho_{k_1}}$ and $\Pi_{\rho_{k_2}}$ in which these cycles $\gamma_{\rho_{k_1}}$ and $\gamma_{\rho_{k_2}}$ exist and are attracting are bounded by the curves obtained from the conditions:

for $\Pi_{\frac{2+2k_1}{5+4k_1}}$:

$$b_1^{(+)} : (f_1 f_2)^{2k_1} f_1 f_2^2 f_1 f_3(1) = 1,$$

$$b_1^{(-)}: f_3 f_1 (f_2 f_1)^{2k_1} f_2^2 f_1 (-1) = -1,$$

$$l^{2k_1+3} p^{2k_1+2} = 1;$$

for $\Pi_{\frac{1+2k_2}{3+4k_2}}$:

$$b_2^{(+)}: (f_1 f_2)^{2k_2} f_2 f_1 f_3 (1) = 1,$$

$$b_2^{(-)}: f_3 (f_1 f_2)^{2k_2} f_2 f_1 (-1) = -1,$$

$$l^{2k_2+2} p^{2k_2+1} = -1;$$

iii) when parameter point (l, p, b) leaves the region $\Pi_{\frac{1+2k_1}{3+4k_1}}$ crossing curve $b_1^{(+)}$ or $b_1^{(-)}$, border-collision bifurcation takes place; it results in a cycle of $(5 + 4k_1)$ invariant intervals at every of which the map f^{5+4k_1} is a skew tent map with the slopes of linear branches equal to $l^{2k_1+3} p^{2k_1+2}$ and $l^{2k_1+2} p^{2k_1+3}$;

iv) when parameter point (l, p, b) leaves the region $\Pi_{\frac{1+2k_2}{3+4k_2}}$ crossing curve $b_2^{(+)}$ or $b_2^{(-)}$, an inverse saddle-node bifurcation takes place; it results in appearance of everywhere dense in $[-1, 1]$ trajectories of $f_{l,p,b}$.

4. Kneading sequences: principal Lemma. Proof of the theorem is based on the theory of kneading sequences of the skew tent map, and firstly, we remind some notions and definitions of this theory to be used below (for detailed see [6, 11 – 13]).

Let f be a unimodal map (Fig. 1, b). We associate with x a finite or infinite sequence of the symbols L, C, R , called *itinerary*. Let define the i th address of a point x $\theta_i(x)$ as follows:

$$\theta_i(x) = \begin{cases} L, & \text{if } f^i(x) \in [0, c); \\ C, & \text{if } f^i(x) = c; \\ R, & \text{if } f^i(x) \in (c, 1], \end{cases} \quad (7)$$

$$i = 0, 1, \dots$$

We define the itinerary of x denoted by $\theta_f(x)$ as follows:

1) $\theta_f(x) = \theta_0(x)\theta_1(x)\dots\theta_n(x)\dots$, if $\theta_i(x) \in \{L, R\}$ for all $i \geq 0$;

2) $\theta_f(x) = \theta_0(x)\theta_1(x)\dots\theta_n(x)\dots$, if $\theta_n(x) = C$ and $\theta_i(x) \in \{L, R\}$ for all $i \in \{0, 1, \dots, n-1\}$.

The sequences $\theta_f(f(c)) = \theta_f(1)$ is called the *kneading sequences* of f .

Let $\underline{A} = A_0 A_1 \dots$ be a sequences of elements $A_i \in \{L, C, R\}$. We say that $\underline{A}$ is *admissible* if one of the following two conditions is satisfied:

1) $\underline{A} = A_0 A_1 \dots A_n \dots$ if $A_i \in \{L, R\}$ for all $i \geq 0$;

2) $\underline{A} = A_0 A_1 \dots A_n$ if $A_n = C$ and $A_i \in \{L, R\}$ for all $i \in \{0, 1, \dots, n-1\}$.

Now, we introduce an ordering in the set of all admissible sequences. We set $L < C < R$ and we extend this ordering to the set of all admissible sequences as follows. Let $K_0 K_1 \dots K_n$ be a finite (or empty) sequence of symbols L, R . We say that $K_0 K_1 \dots K_n$ is even (odd) if it has an even (odd) number of R 's. Assume that $\underline{K} = K_0 K_1 \dots$ and $\underline{K}' = K'_0 K'_1 \dots$ are admissible sequences such that $\underline{K} \neq \underline{K}'$. Let n be such that $K_i = K'_i$ for $i < n$ and $K_n \neq K'_n$. Then we say that $\underline{K} < \underline{K}'$ if either

1) $K_n < K'_n$ and $K_0 K_1 \dots K_{n-1}$ is even;

2) $K_n > K'_n$ and $K_0 K_1 \dots K_{n-1}$ is odd.

The following result about monotonicity of kneading sequences for skew tent map was obtained in [11, 12]. Consider the map $f = f_{l,p}$ in the form (3) and let $(l, p) \in D_n = \{-\frac{1-l^n}{l^{n-1}(1-l)} < p \leq -\frac{1}{l^{n-1}}\}$, $n = 2, 3, \dots$. Denote $K(l, p)$ a kneading sequences for this map. If $(l', p') > (l, p)$ (i. e. $l' \geq l$ $p' \geq p$ and at least one of these inequalities is strict) then $K(l'p') > K(l, p)$.

To prove the Theorem, we need the following Lemma.

Lemma. For the skew tent map $f_{l,p}$ ($0 < l < 1$, $p < -1$) in the window between the regions $\Pi_{1/2}$ and $\Pi_{1/3}$, there exist the only two sequences of the cycles $\gamma_{(2+2k_1)/(5+4k_1)}$ and $\gamma_{(1+2k_2)/(3+4k_2)}$ $k_1, k_2 = 0, 1, \dots$ with kneadings $RLR(RLRL)^i RC$ and $RLRRL(RLRL)^i RC$ $i = 0, 1, \dots$, such that the number of symbols R in each of them is not greater than the number of symbols L plus number of symbols C plus 1.

Proof. Consider the skew tent map $f_{l,p}$ in the form (3). Using the property of monotonicity of kneading sequences for skew tent map, let's construct kneading sequences between $RLRC$ and RLC corresponding to the cycles $\gamma_{2/4}$ and $\gamma_{1/3}$.

Step 1. The first two symbols in kneading sequence are obviously R and then L . For third symbol, generally, three possibilities exist: R , C and L . However, ordering for third symbol is: $R < C < L$ (the number of symbols R before third symbols is odd) therefore, third symbol should be R . So, we have:

$$RLRC < RLR... < RLC.$$

Note, that combinations LL and LC are not admissible in the window between $RLRC$ and RLC . For forth symbol the ordering is a contrary: $L < C < R$, hence the forth symbol will be R , and we obtain:

$$RLRC < RLRR... < RLC.$$

Further, in the fifth place it can be any of three symbols L , C and R ordering for which is $R < C < L$. So, we have three possibilities for the continuation of the kneading $RLRR...$ and then ordered by:

$$RLRC < RLRRR... < RLRRC < RLRRL... < RLC.$$

Kneading $RLRRR...$, at arbitrary continuation, has the number of the symbols R more them common number of symbols L and C plus 1 (the combinations LL and LC are not possible in this window) which contradicts to the condition of Lemma. Thus, $RLRRC$ is the first (from the left side) admissible kneading in the window considered.

Step 2. Consider the next kneading $RLRRL...$. At the sixth place in it, the only symbol R can be put (combinations LC and LL are not admissible), but the seventh symbol may be any from L , C and R for which the order is $L < C < R$. So, we have:

$$RLRC < RLRRC < RLRRRL... < RLRRRLC < RLRRRLR... < RLC.$$

Analogically to the step 1 we conclude that the kneading $RLRRRLR...$ doesn't satisfy the conditions of the Lemma at the arbitrary its continuation.

The eight symbol in the kneading $RLRRRLR...$ can be the only R , meantime any symbol from L , C and R may be put at the ninth place. The order for the ninth symbol is $R < C < L$. So, we obtain:

$$RLRC < RLRC < RLRLRLRR... < RLRLRLRC < \\ < RLRLRLRL... < RLRLRC < RLC.$$

The kneading $RLRLRLRR...$ doesn't satisfy the condition of the Lemma. Therefore, by step 2 we have constructed two kneadings $RLRLRLRC$ and $RLRLRC$ which are admissible according to the conditions of the Lemma. Moreover, $RLRLRLRC$ is the first one larger then $RLRC$ (being constructed in the step 1) and $RLRLRC$ is the first one smaller then the maximal kneading of the window RLC . Therefore,

$$RLRC < RLRC < RLRLRLRC < RLRLRLRL... < RLRLRC < RLC.$$

Step 3. Repeating the arguments of the step 2, we obtain two more kneadings satisfying the conditions of the Lemma and ordered as follows:

$$RLRC < RLRC < RLRLRLRC < RLRLRLRLRLRC < \\ < RLRLRLRLRLRL... < RLRLRLRLRC < RLRLRC < RLC.$$

And so on. After $(i - 1)$ steps we obtain the sequence of kneadings satisfying the conditions of the Lemma and ordered as follows:

$$RLRC < RLRC < ... < RLRL(RLRL)^{i-1}RLRC < \\ < RLRL(RLRL)^iRLRC < RLRL(RLRL)^{i+1} < ... \\ ... < RLRL(RLRL)^iRC < RLRL(RLRL)^{i-1}RC < ... < RLC.$$

It follows from the construction of the sequence that there are not any other kneadings satisfying conditions of the Lemma in the window considered. Therefore, in this window the following order occurs:

$$RLRC < RLRC < ... < RLRL(RLRL)^{i-1}RLRC < \\ < RLRL(RLRL)^iRLRC < ... < RLR(RL)^\infty < ... < \\ < RLRL(RLRL)^iRC < RLRL(RLRL)^{i-1}RC < ... < RLC.$$

Lemma is proved.

5. Proof of the Theorem. Let's consider map (1) at $b = 1 + 1/p$ i.e. parameter point (l, p, b) belongs to surface of the transition „from unimodal map to bimodal map”. Our map is unimodal (skew tent map) at this moment on the invariant interval (Fig. 1, b). Let parameter l be any fixed and we consider a part of the curve $b = 1 + 1/p$ between the regions $\Pi_{1/2}$ and $\Pi_{1/3}$. From one side this curve is tangent to the angle point between boundaries of the regions $\Pi_{1/2}$ and $\Pi_{2/4}$ and from the other side it is tangent to the boundary of the region $\Pi_{1/3}$. Point cycles $\gamma_{2/4}$ at $(l, p) \in \Pi$ and $\gamma_{1/3}$ at $(l, p) \in \Pi$ for bimodal map $f_{l,p,b}$ and their symbolic sequences are $LMRL...$ and $LLR...$ correspondingly.

Meantime our map may be considered as a unimodal map on the invariant interval on the arc of curve $b = 1 + 1/p$. Then the kneadings of that cycles on the ends of the arc equal $RLRC$ and RLC .

According to the Lemma, there exist two sequences of kneading, which correspond to some point cycles (for skew tent map):

$$RLRC < RLRRC < \dots < RLRRL(RLRL)^i RLRC < \dots < \\ < RLR(RL)^\infty < \dots < RLRRL(RLRL)^i RC < \dots < RLC, \quad i = 0, 1, \dots$$

Note, that these sequences have the same limit kneading at $i \rightarrow \infty$ which equals $RLR(RL)^\infty$.

Let the parameter point (l, p, b) is crossing the surface $b = 1 + 1/p$ and enters the regions Π . Then our map becomes bimodal map on the invariant interval (i.e. right branch of the map appears). So, according to continuity, the last right point of the cycle continuously passes into right branch of the bimodal map.

Therefore, in this window the two sequences of the point cycles appear in neighborhood curve with corresponding symbolic sequences

$$RLMR < RLMML < \dots < RLMML(ML)^{2i} < RLMML(MLML)^i \dots \\ \dots < RLMML(ML)^{2i-1} < \dots < RLL, \quad i = 1, 2, \dots$$

Knowing the symbolic sequence for a cycle, it's easy to define the rotation number of this cycle: for each symbolic sequences there corresponds the only one generalization rotation number. So, between regions $\Pi_{1/2}$ and $\Pi_{1/3}$ in this window there exist two sequences of the point cycles with corresponding rotation number and symbolic sequences:

$$\rho_{k_1} = \frac{2 + 2k_1}{5 + 4k_1}, \quad (LM)^{2k_1} LM^2 LR \dots, \quad k_1 = 0, 1, \dots,$$

$$\rho_{k_2} = \frac{1 + 2k_2}{3 + 4k_2}, \quad (LM)^{2k_2} MLR \dots, \quad k_2 = 0, 1, \dots$$

i) Let's now consider stability of these cycles. Their multipliers are:

$$\mu_{k_1} = l^{2k_1+3} p^{2k_1+2} = l(lp^{2k_1+2}), \quad \mu_{k_2} = l^{2k_2+2} p^{2k_2+1} = l(lp^{2k_2+1}).$$

Note, that $|lp| > 1$ in this part of the parameter space. Hence, $0 < \mu_{k_1} < 1$ and $-1 < \mu_{k_2} < 0$ if $l > 0$ is small enough. Moreover, for any $l \in (0, 1)$ there exist integers m_1 and m_2 such that these inequalities are fulfilled if and only if $k_1 < m_1$ and $k_2 < m_2$. In other words: only the first m_1 and m_2 cycles of these sequences are attracting. It's easy to obtain that $m_1 = [ln(l^3 p^2)^{-1/2}] + 1$, and $m_2 = [ln|l^2 p|^{-1/2}] + 1$. So, $m_1 \rightarrow \infty$ and $m_2 \rightarrow \infty$ at $l \rightarrow 0$.

ii) Knowing the symbolic sequences of these cycles, the conditions of their existence and stability can be formulated with using the fact that the boundaries of their regions are obtained from the condition of border-collision, saddle node or flip bifurcations. After some transformation, formulas for the boundaries of these regions can be represented as follows:

for $\Pi_{\frac{2+2k_1}{5+4k_1}}$:

$$p = -l^{\frac{2k_1+3}{2k_1+2}},$$

$$b_1^{(-)} = \frac{(l^{2k_1+3} p^{2k_1+2} - 1)(1 - l^2 p^2)}{l[(lp)^{2k_1}(1 + p(1 + p) + lp^2(1 + l))(1 - l^2 p^2) + (1 + p)(1 + lp)(1 - (lp)^{2k_1})]} +$$

$$+ \frac{(1-l/p)[(lp)^{2k_1}(1+lp^2(1-l))(1-l^2p^2) + (1+lp)(1-(lp)^{2k_1})]}{l[(lp)^{2k_1}(1+p(1+p)+lp^2(1+l))(1-l^2p^2) + (1+p)(1+lp)(1-(lp)^{2k_1})]},$$

$$b_1^{(+)} = \frac{(1-l^{2k_1+3}p^{2k_1+2})(1-l^2p^2)}{l[l^{2k_1+1}p^{2k_1}(1+p(1+p)+lp^2+1)(1-l^2p^2) + l(1+p)(1+lp)(1-(lp)^{2k_1})]} +$$

$$+ \frac{(1-l/p)[(l^{2k_1+1}p^{2k_1}(1+lp^2)-1)(1-l^2p^2) + l(1+lp)(1-(lp)^{2k_1})]}{l[l^{2k_1+1}p^{2k_1}(1+p(1+p)+lp^2+1)(1-l^2p^2) + l(1+p)(1+lp)(1-(lp)^{2k_1})]};$$

for $\Pi_{\frac{1+2k_2}{3+4k_2}}:$

$$p = -l^{\frac{2k_2+2}{2k_2+1}},$$

$$b_2^{(-)} = \frac{(1-l^{2k_2+2}p^{2k_2+1})(1-l^2p^2)}{l[(1+l^{2k_2}p^{2k_2+1}(1+l))(1-l^2p^2) + l(1+p)(1+lp)(1-(lp)^{2k_2})]} +$$

$$+ \frac{(1-l/p)[(lp)^{2k_2+1}-1)(1-l^2p^2) + l(1+lp)(1-(lp)^{2k_2})]}{l[(1+l^{2k_2}p^{2k_2+1}(1+l))(1-l^2p^2) + l(1+p)(1+lp)(1-(lp)^{2k_2})]},$$

$$b_2^{(+)} = \frac{(l^{2k_2+2}p^{2k_2+1}-1)(1-l^2p^2)}{l^{2k_2}p^{2k_2+1}(1+l(1+l))(1-l^2p^2) + l(1+p)(1+lp)(1-(lp)^{2k_2})} +$$

$$+ \frac{(1-l/p)[l^{2k_2}p^{2k_2+1}(1-l)(1-l^2p^2) + (1+lp)(1-(lp)^{2k_2})]}{l^{2k_2}p^{2k_2+1}(1+l(1+l))(1-l^2p^2) + l(1+p)(1+lp)(1-(lp)^{2k_2})}.$$

These formulas have been used when obtaining Fig. 2.

iii) When parameter point (l, p, b) intersects the boundary curves $b_1^{(+)}$ or $b_1^{(-)}$ of any region $\Pi_{\frac{1+2k_2}{3+4k_1}}$, border-collision bifurcation takes place. Indeed, so as $\mu_{k_1} > 0$ result of the bifurcation is the birth of a cycle of $(5+4k_1)$ infinitesimal invariant intervals, at every of which the map f^{5+4k_1} is a skew tent map with the slopes of linear branches equal $l^{2k_1+3}p^{2k_1+2}$ and $l^{2k_1+2}p^{2k_1+3}$. Thus, in this a case the bifurcation results in attracting point cycle or cycle of chaotic intervals [4,7].

iv) When parameter point (l, p, b) intersects the boundary curves $b_2^{(+)}$ or $b_2^{(-)}$ of the region $\Pi_{\frac{1+2k_2}{3+4k_2}}$, an inverse saddle-node bifurcation takes place. Indeed, so as $\mu_{k_2} < 0$ after this bifurcation we can't obtaine any finitesimal invariant intervals of the map f^{3+4k_2} in the neighborhood of $\gamma_{\frac{1+2k_2}{3+4k_2-2}}$.

Theorem is proved.

6. Remarks. 1. At $k_1, k_2 \rightarrow \infty$ the regions for the cycles γ_{k_1} and γ_{k_2} become more dense approaching the surfaces of homoclinic bifurcations of the cycle $\gamma_{1/2}$ with rotation number $1/2$. The moments of homoclinic bifurcation are determined from the conditions

$$f_2 f_1 f_3(1) = x_2,$$

$$f_2 f_1(-1) = x_2,$$

where $x_2 = f_1 f_2(x_2)$. Therefore, formulas for the boundaries are as follow:

$$b_{col}^- = \frac{(lp-1)((l-p)(1-l)+p^2)+(1-l/p)}{l(1+p)+p(1+l+l^2)(lp-1)};$$

$$b_{col}^+ = \frac{(p-l-lp)(lp-1)+(1-l/p)}{p(1+l)(lp-1)+l(1+p)}.$$

2. Neighboring regions for the cycles of second sequence $\gamma_{1+2k_2/3+4k_2}$, $k_2 = 0, 1, \dots$ can intersect each other. It means that multistability phenomenon takes place: in each of these intersecting regions, two attracting cycles coexist [9].

3. As numerical evidence suggests, there exist six period adding sequences of attracting cycles in the window between regions $\Pi_{1/3}$ and $\Pi_{1/4}$. Moreover, between any two neighboring parameter regions of the one of these sequences there exist two more embedded period adding sequences [10]. Note, finally, that bifurcation picture becomes more and more complicated in the next window.

1. Sharkovsky A.N., Maistrenko Yu., Deregel Ph., Chua L.O. Dry turbulence from a time-delayed Chua's circuit //J. Circuits, Systems and Computers. - 1993. - 3, № 2. - P. 645 - 668.
2. Mira Ch., Gardini L., Barugola A., Cathala J.C. Chaotic dynamics in two-dimensional noninvertible maps //World Sci. Ser. A. - 1996. - 20. - 607 p.
3. Maistrenko V.L., Maistrenko Yu.L., Sushko I.M. Attractors of piecewise linear maps of straight line and plane. - Kiev, 1992. - 55 p. - (Preprint/ Nat. Acad. Sci. Ukraine. Inst. Math.; № 92-33).
4. Maistrenko Yu.L., Maistrenko V.L., Chua L.O. Cycles of chaotic intervals in a time-delayed Chua's circuits //Int. J. Bifurcation and Chaos. - 1993. - 3, № 6. - P. 1557 - 1572.
5. Nusse H.E., Yorke J.A. Border-collision bifurcations for piecewise smooth one-dimensional maps. //Ibid. - 1995. - 5, № 1. - P. 189 - 207.
6. de Melo W., van Strien S. One-dimensional dynamics. - Berlin etc.: Springer-Verlag, 1993. - 605 p.
7. Maistrenko Yu.L., Maistrenko V.L., Vikul S.I., Chua L.O. Bifurcation of attracting cycles from time-delayed Chua's circuit //Int. J. Bifurcation and Chaos. - 1995. - 5, № 3. - P. 653-671.
8. Maistrenko Yu.L., Maistrenko V.L., Vikul S.I. Bifurcation of attracting cycles of piecewise linear interval maps //J. Tech. Phys. - 1996. - 37, № 3-4. - P. 367 - 370.
9. Maistrenko Yu.L., Maistrenko V.L., Vikul S.I. Transition from unimodal to bimodal maps in one-dimensional piecewise linear models //Proc. Int. Conf. ICNBP'96 (Lodz, Poland, 16-18 Sept. 1996). - Lodz, 1996. - P. 173 - 179.
10. Maistrenko Yu.L., Maistrenko V.L., Vikul S.I. On period-adding sequences of attracting cycles in piecewise linear maps //J. Chaos, Solitons and Fractals. - 1998. - 9, № 1/2. - P. 67 - 75.
11. Misiurewicz M., Visinescu E. Kneading sequences of skew tent maps //Ann.Inst. H. Poincaré. - 1991. - 27, № 1. - P. 125 - 140.
12. Marcuard J.C., Visinescu E. Monotonicity properties of some skew tent maps //Probab. Statist. - 1992. - 28, № 1. - P. 1 - 29.
13. Sharkovsky A.N., Maistrenko Yu.L., Romanenko E.Yu. Difference equations and their applications. - Kluwer Acad. Publ., 1993. - 358 p.

Received 04.05.98