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THE LIE-ALGEBRAIC STRUCTURE OF LAX TYPE INTEGRABLE NONLOCAL DIFFERENTIAL-DIFFERENCE EQUATIONS

ЛІ-АЛГЕБРАЇЧНА СТРУКТУРА НЕЛОКАЛЬНИХ ДИФЕРЕНЦІАЛЬНО-РІЗНИЦЕВИХ РІВНЯНЬ, ІНТЕГРОВНИХ ЗА ЛАКСОМ

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Some centrally extended current operator Lie algebra is considered as a symmetry Lie algebra of the Lax type integrable nonlocal differential-difference dynamical systems. As a result of its dimerization, we obtain the Lax type representations for some local differential-difference equations as well as for some nonlocal ones. An alternative approach to the Lie-algebraic interpretation of the integrable local differential-difference systems is proposed.

Розглядається деяка центрально розширена операторна алгебра Лі петель як алгебра Лі симетрій інтегровних за Лаксом нелокальних диференціально-різницевих динамічних систем. В результаті її двомеризації отримано зображення Лакса для деяких локальних та нелокальних диференціально-різницевих рівнянь. Запропоновано альтернативний підхід до Лі-алгебраїчної інтерпретації інтегровних локальних диференціально-різницевих систем.

1. Introduction. Let $\mathcal{A}$ be an arbitrary continuous associative algebra over **C**, $T : \mathcal{A} \to \mathcal{A}$ — some its automorphism:

$$T((k_1a_1 + k_2a_2) = k_1T(a_1) + k_2T(a_2), \quad T(ab) = T(a)T(b)$$

for all $a_1,\ a_2 \in \mathcal{A}$ and $k_1,\ k_2 \in \mathbf{C}$. Assume also that on the algebra $\mathcal{A}$ there is defined a linear mapping $\tau : \mathcal{A} \to \mathbf{C}$ with such properties:

$$\tau(ab) = \tau(ba), \quad \tau(Ta) = \tau(a)$$

for $a, b \in \mathcal{A}$. Define now an algebra $\mathcal{G}$ of all linear mapping $A : \mathcal{A} \to \mathcal{A}$ having such a structure: $A \in \mathcal{G}$ iff

$$A(T) = \sum_{i<<\infty} a_i T^i, \tag{1}$$

where $a_i \in \mathcal{A}$ and $i \in \mathbf{Z}$. On the associative operator algebra $\mathcal{G}$ one can define [1] the following linear trace-operation:

$$\mathrm{tr}\, A(T) := \tau(a_0),$$

satisfying such additional important relationships:

$$\mathrm{tr}\,(A(T)B(T)) = \mathrm{tr}\,(B(T)A(T)) \tag{2}$$

for all $A(T),\ B(T) \in \mathcal{G}$. It is the standard procedure of converting the associative algebra $\mathcal{G}$ into a Lie algebra $\mathcal{G}$ with the Lie product chosen the commutator in $\mathcal{G}$:

$$[A(T), B(T)] := A(T)B(T) - B(T)A(T) \tag{3}$$

for $A(T)$ and $B(T) \in \mathcal{G}$. In addition we also assume here that the scalar product

$$\langle A(T), B(T)\rangle := \mathrm{tr}\,(A(T)B(T)) \tag{4}$$

is nondegenerate on $\mathcal{G}$, that is, from the condition $\langle A(T), B(T)\rangle = 0$ for a fixed $A(T) \in \mathcal{G}$ and $B(T)$, it follows that $A(T) \equiv 0$.

Thus we have constructed above the operator Lie algebra $\mathcal{G}$ (1) and (3) which is endowed with the nondegenerate scalar product (4), which is owing to (2) symmetric and *ad*-invariant, that is

$$\langle A(T), [B(T), C(T)]\rangle = \langle [A(T), B(T)], C(T)\rangle$$

for any $A(T),\ B(T)$ and $C(T) \in \mathcal{G}$.

2. Central extension. Assume now that the basic associative algebra $\mathcal{A}$ depends effectively on an independent parameter $x \in \mathbf{S}^1$, which makes it possible to define the new associative current algebra $C^1_{\mathbf{S}}(\mathcal{A}) = C^\infty(\mathbf{S}^1; \mathcal{A})$ which naturrally generates [2] via the construction above the corresponding current operator Lie algebra $C^1_{\mathbf{S}}(\mathcal{G}) = C^\infty(\mathbf{S}^1; \mathcal{G})$ with the following modified Tr-operation:

$$\mathrm{Tr}\, A(T) := \int_{\mathbf{S}^1} dx\, \tau(a_0)(x)$$

for any $A(T) \in C^1_{\mathbf{S}}(\mathcal{G})$. It is obvious that on the current Lie algebra $C^1_{\mathbf{S}}(\mathcal{G})$ there is defined the corresponding to (4) scalar product:

$$(A(T), B(T)) := \mathrm{Tr}\,(A(T)B(T)) \tag{5}$$

for $A(T)$ and $B(T) \in C^1_{\mathbf{S}}(\mathcal{G})$.

The current Lie algebra $C^1_{\mathbf{S}}(\mathcal{G})$ can be naturally extended now via its central extension procedure: $C^1_{\mathbf{S}}(\mathcal{G}) :\to C^1_{\mathbf{S}}(\hat{\mathcal{G}}) = (C^1_{\mathbf{S}}(\mathcal{G}) \oplus \mathbf{C})$ with such a Lie product:

$$[(A(T), \alpha), (B(T), \beta)] := ([A(T), B(T)], \omega_2(A(T), B(T))), \tag{6}$$

where by definition, $\omega_2 : C^1_{\mathbf{S}}(\mathcal{G}) \times C^1_{\mathbf{S}}(\mathcal{G}) \to \mathbf{C}$ is the standard Maurer – Cartan 2-cocycle on $C^1_{\mathbf{S}}(\mathcal{G})$, $\alpha, \beta \in \mathbf{C}$ and for any $A(T)$, $B(T) \in C^1_{\mathbf{S}}(\mathcal{G})$

$$\omega_2(A(T), B(T)) := \int_{\mathbf{S}^1} dx\, \langle A(T), dB(T)/dx \rangle. \tag{7}$$

Assume now, in addition that an automorphism $T : \mathcal{A} \to \mathcal{A}$ is naturally extended to an automorphism $T : C^1_{\mathbf{S}}(\mathcal{A}) \to C^1_{\mathbf{S}}(\mathcal{A}) \to \mathbf{C}$. As a result we have constructed the centrally extended current operator Lie algebra $C^1_{\mathbf{S}}(\hat{\mathcal{G}})$ which we shall interpret further as a symmetry Lie algebra of specially built integrable nonlocal differential-difference dynamical systems.

3. $\mathcal{R}$-matrix method. Consider now a nonlinear smooth dynamical system

$$\frac{du}{dt} = K[u] \tag{8}$$

on a 2π-periodic functional Poisson manifold $M \subset C^\infty(\mathbf{S}^1; \mathbf{R}^m)$, where $K : M \to T(M)$ — the corresponding vector field on M. Assuming that dynamical system (8) possesses a symmetry subalgebra isomorphic to some current Lie subalgebra $C^1_{\mathbf{S}}(\hat{\mathcal{G}}_0)$, define also apriori some momentum mapping $\hat{l} : M \to C^1_{\mathbf{S}}(\hat{\mathcal{G}}^*)$ associated with the corresponding current Lie algebra action of $C^1_{\mathbf{S}}(\hat{\mathcal{G}})$ on M, being considered Poissonian [3], that is the Poisson structure $\{.,.\}$ on M is invariant with respect to this action and the diagram:

$$\begin{array}{ccc} M & \xrightarrow{\hat{l}} & C^1_{\mathbf{S}}(\hat{\mathcal{G}}^*) \\ A \downarrow & & \downarrow ad^*_{A^{-1}} \\ M & \xrightarrow{\hat{l}} & C^1_{\mathbf{S}}(\hat{\mathcal{G}}^*) \end{array} \tag{9}$$

is commuting for any $A \in C^1_{\mathbf{S}}(\hat{\mathcal{G}})$. As a result of the construction above one can represent the Poisson structure on the manifold M as the standard reduction procedure of the canonical Lie – Poisson structure on the adjoint space $C^1_{\mathbf{S}}(\hat{\mathcal{G}}^*)$:

$$\{\gamma, \mu\} := (\hat{l}, [\nabla\gamma(\hat{l}), \nabla\mu(\hat{l})])|_{\hat{l}:=\hat{l}[u]} \tag{10}$$

for $u \in M$ and any smooth functionals $\gamma, \mu \in \mathcal{D}(C^1_{\mathbf{S}}(\hat{\mathcal{G}}^*))$. Concerning the integrable Poissonian flows (8) on M there needs to construct a hierarchy of Poisson commuting functionals $h_j \in \mathcal{D}(M)$, $j \in \mathbf{Z}_+$, which we shall produce here via the standard $\mathcal{R}$-matrix [2] approach. Namely, consider the space $I(C_{\mathbf{S}^1}(\hat{\mathcal{G}}^*))$ of Casimir functionals $\gamma \in \mathcal{D}(C_{\mathbf{S}^1}(\hat{\mathcal{G}}^*))$, satisfying due to (10) and (6) such a functional equation:

$$\frac{d\nabla\gamma(l)}{dx} = [l, \nabla\gamma(l)] \tag{11}$$

for all $\hat{l} := (l, 1) \in C_{\mathbf{S}^1}(\hat{\mathcal{G}}^*)$. Assume also now that the current Lie algebra $C_{\mathbf{S}^1}(\hat{\mathcal{G}})$ admits a standard $\mathcal{R}$-structure [2], that is the new bracket

$$[A, B]_{\mathcal{R}} := [A, \mathcal{R}B] + [\mathcal{R}A, B],$$

defined by means of a linear homomorphism $\mathcal{R} : C_{\mathbf{S}^1}(\hat{\mathcal{G}}) \to C_{\mathbf{S}^1}(\hat{\mathcal{G}})$, is a Lie bracket too for all $A, B \in C_{\mathbf{S}^1}(\hat{\mathcal{G}})$. This is the case [2] if a homomorphism $\mathcal{R} : C_{\mathbf{S}^1}(\hat{\mathcal{G}}) \to C_{\mathbf{S}^1}(\hat{\mathcal{G}})$ satisfies the Yang – Baxter condition:

$$\mathcal{R}[A, B]_{\mathcal{R}} = \frac{1}{4}[A, B] + [\mathcal{R}A, \mathcal{R}B] \tag{12}$$

for any $A, B \in C_{\mathbf{S}^1}(\hat{\mathcal{G}})$. In particular, if $\mathcal{R} \in \mathrm{Hom}\,(C^1_{\mathbf{S}}(\mathcal{G}))$, then the corresponding mapping $\mathcal{R} \in \mathrm{Hom}\,(C_{\mathbf{S}^1}(\hat{\mathcal{G}}))$ one gets naturally as

$$[(A, \alpha), (B, \beta)]_{\mathcal{R}} := ([A, B], \omega_2(\mathcal{R}A, B) + \omega_2(A, \mathcal{R}B))$$

for any $A, B \in C_{\mathbf{S}^1}(\hat{\mathcal{G}})$ and $\alpha, \beta \in \mathbf{C}$. Now we are in a position to formulate the following theorem.

Theorem 1. *Given a hierarchy of Casimir functionals $\gamma_j \in (C_{\mathbf{S}^1}(\hat{\mathcal{G}}^*))$, $j \in \mathbf{Z}_+$, solving the equation (11), then all reduced on M functionals $h_j := \gamma_j|_{l=l[u]}$, $u \in M$, $j \in \mathbf{Z}_+$, are in involution with respect to the Poisson bracket*

$$\{\gamma, \mu\}_{\mathcal{R}} := (l, [\nabla\gamma(l), \nabla\mu(l)]_{\mathcal{R}})|_{l:=l[u]} \tag{13}$$

reduced from $C_{\mathbf{S}^1}(\hat{\mathcal{G}}^*)$ on M with respect to the diagram (9). Moreover, due to bracket (13) the momentum mapping $\hat{l} : M \to C_{\mathbf{S}^1}(\hat{\mathcal{G}}^*)$ satisfies the following evolution equation in $C^1_{\mathbf{S}}(\mathcal{G}^*)$:

$$\frac{dl}{dt} = ad^*_{\mathcal{R}\nabla\gamma(l)}l - \frac{d(\mathcal{R}\nabla\gamma(l))}{dx}, \tag{14}$$

where $\gamma \in I(C_{\mathbf{S}^1}(\hat{\mathcal{G}}^*))$ is Hamiltonian function, $t \in \mathbf{R}$ — an evolution parameter. The flow (14) is equivalent to the Lax type representation of dynamical system (8):

$$\frac{dl}{dt} = \left[l - \frac{d}{dx}, \mathcal{R}\nabla\gamma(l)\right], \tag{15}$$

owing to the supposed nondegeneracy of the scalar product (5) on $C^1_{\mathbf{S}}(\mathcal{G}^*) \simeq C^1_{\mathbf{S}}(\mathcal{G})$.

The proof of this theorem is fulfilled via the standard scheme devised in [4, 5].

4. Integrable nonlocal differential-difference dynamical systems. Assume now that the algebra $C^1_{\mathbf{S}}(\mathcal{G})$ automorphism $T : C^1_{\mathbf{S}}(\mathcal{G}) \to C^1_{\mathbf{S}}(\mathcal{G})$ is a simple shift on $\delta \in \mathbf{S}^1$ along the parameter $x \in \mathbf{S}^1$:

$$T \circ a(x) := a(x + \delta \mathrm{i}), \ \ x \in \mathbf{S}^1, \ \ \mathrm{i}^2 = -1,$$

for any element $a \in C^1_{\mathbf{S}}(\mathcal{G})$, and consider for simplicity $\mathcal{A} = \mathbf{R}$.

It is evident that the corresponding space $C^1_{\mathbf{S}}(\mathcal{G})$ will be an operator Lie algebra with the nondegenerate scalar product (5), where by definition, we put $\tau(a) = a$ for any $a \in \mathbf{R}$. Observed further that the current Lie algebra $C^1_{\mathbf{S}}(\mathcal{G})$ admits the usual splitting into two subalgebras:

$$C^1_{\mathbf{S}}(\mathcal{G}) = C^1_{\mathbf{S}}(\mathcal{G}_+) \oplus C^1_{\mathbf{S}}(\mathcal{G}_-),$$

where by definition,

$$C^1_{\mathbf{S}}(\mathcal{G}_+) := \left\{ \sum_{i \in \mathbf{Z}_+}^{i << \infty} a_i T^i \in C^1_{\mathbf{S}}(\mathcal{G}) : a_i \in C^1_{\mathbf{S}}(\mathbf{R}) \right\},$$
$$C^1_{\mathbf{S}}(\mathcal{G}_-) := \left\{ \sum_{i \in \mathbf{Z}_-} T^i \circ a_i \in C^1_{\mathbf{S}}(\mathcal{G}) : a_i \in C^1_{\mathbf{S}}(\mathbf{R}) \right\}, \tag{16}$$

one gets the following partial solution to the Yang – Baxter equation (12):

$$\mathcal{R} := \frac{1}{2}(P_+ - P_-),$$

where $P_\pm : C^1_{\mathbf{S}}(\mathcal{G}) \to C^1_{\mathbf{S}}(\mathcal{G}_\pm)$ – the corresponding projectors on the subalgebras (16). If $A(T)$ and $B(T) \in C^1_{\mathbf{S}}(\mathcal{G})$ are given, then one can calculate that

$$(A(T), B(T)) = \sum_{j \in \mathbf{Z}} \int_{\mathbf{S}^1} a_j(x) b_j(x) dx, \tag{17}$$

where we put

$$A(T) := \sum_{i \in \mathbf{Z}}^{i << \infty} a_i T^i, \quad B(T) := \sum_{j \in \mathbf{Z}}^{j << \infty} T^{-j} \circ b_j \tag{18}$$

for some $a_i, b_j \in C^1_{\mathbf{S}}(\mathbf{R}), i, j \in \mathbf{Z}$. As a result of the formula (17) one can identify spaces $C^1_{\mathbf{S}}(\mathcal{G}^*)$ and $C^1_{\mathbf{S}}(\mathcal{G})$, with the relationships

$$C^1_{\mathbf{S}}(\mathcal{G}^*_+) \simeq T \circ C^1_{\mathbf{S}}(\mathcal{G}_-), \quad C^1_{\mathbf{S}}(\mathcal{G}^*_-) \simeq C^1_{\mathbf{S}}(\mathcal{G}_+) \circ T,$$

being held on $C^1_{\mathbf{S}}(\mathcal{G}^*)$. Thus one can now simply generate Lax type Hamiltonian systems via the recipe (15), where $\gamma \in I(C_{\mathbf{S}^1}(\hat{\mathcal{G}}^*))$ satisfies the equation (11).

Example. Let us take an element $l \in C^1_{\mathbf{S}}(\mathcal{G}^*_+)$ as

$$l := l[u] = \mathrm{i}\alpha T + \mathrm{i}\varepsilon^{-1} u, \tag{19}$$

where $\alpha \in \mathbf{C}$ and $\varepsilon \in \mathbf{R}$ – some constants, $u \in M \subset C^\infty(\mathbf{S}^1; \mathbf{R})$.

The equation (11) can be simply enough solved putting for each $n \in \mathbf{Z}_+$

$$\nabla \gamma_n(l) := \sum_{n-j \in \mathbf{Z}_+} a_j T^j, \tag{20}$$

where $a_j \in C^\infty(\mathbf{S}^1; \mathbf{R}), n - j \in \mathbf{Z}_+$ – some unknown functional parameters which are obtained from (11) successively via the recurrent procedure. As a result one gets that at $n = 2$

$$a_2 = \mathrm{i}\varepsilon\alpha^2, \quad a_1 = \mathrm{i}\alpha(Tu + u + 2u\varepsilon\delta^{-1}),$$
$$a_0 = (H + 2\mathrm{i}\delta^{-1}\partial_x^{-1})u_x + \mathrm{i}\varepsilon^{-1}u^2, \ldots, \tag{21}$$

where

$$H := (T + 1)(T - 1)^{-1}$$

— the well known integral operator acting as

$$H : f \in \mathcal{S}(\mathbf{R}; \mathbf{C}) = -\frac{\mathrm{i}}{\delta}\int\limits_{\mathbf{R}} \operatorname{cth}\frac{\pi(x-\xi)}{\delta} f(\xi)d\xi \in \mathcal{S}(\mathbf{R}; \mathbf{C}),$$

on the Schwarz space $\mathcal{S}(\mathbf{R}; \mathbf{C})$ of the fastly decreasing functions. Taking also into account that the evolution equation (15) can be rewritten equivalently as

$$\frac{dl}{dt} = \left[l - \frac{d}{dx}, (\nabla\gamma(l))_+\right] \tag{22}$$

where $\gamma \in I(C_{\mathbf{S}^1}(\hat{\mathcal{G}}^*))$, from (19) – (21) and (22) one gets immediately the following nonlocal differential-difference Korteweg – de Vries type shallow water equation:

$$\frac{\partial u}{\partial t} = -\mathrm{i}\varepsilon(H + 2\mathrm{i}\delta^{-1}\partial_x^{-1})u_{xx} + 2uu_x, \tag{23}$$

first found in [6, 7] by means of completely different approaches. As $\delta \to 0$, $\varepsilon \doteq 2\delta^{-1}$ the equation (23) appears to reduce into the usual Korteweg – de Vries equation

$$\frac{\partial u}{\partial t} = \frac{1}{3}u_{xxx} + 2uu_x,$$

and alternatively, as $\delta \to \infty$, $\varepsilon = 1$, (23) reduces into the well known Benjamin – Ono deep water equation:

$$\frac{\partial u}{\partial t} = 2uu_x + \frac{1}{\pi}\int\limits_{\mathbf{R}} \frac{u_{\xi\xi}(\xi)d\xi}{\xi - x} \tag{24}$$

where we have assumed that the period in $x \in \mathbf{R}$ of the manifold M has been put to infinity. The latter equation (24) was proved to be Lax type integrable too [4, 5] by means of a differential Riemann – Hilbert problem in a strip of the complex extension of the variable $x \in \mathbf{R}$ between $\operatorname{Im} x = 0$ and $\operatorname{Im} x = \pm 2\delta$, and then periodically extended vertically to infinity.

5. Dimerization of the nonlocal differential-difference equations. Consider now a situation when a functional element $u \in \hat{M} \subset C^\infty(\mathbf{R}; \mathcal{B}^n)$ with $\mathcal{B}$ being some associative algebra, for instance, the algebra of pseudodifferential operators. An automorphism $T : \mathcal{A} \to \mathcal{A}$, where

$\mathcal{A} \subset C^{\infty}(\mathbf{R};\mathcal{B})$, naturally deines the associative subalgebra $\mathcal{G}$ of homomorphism (1) which then is transformed into the operator Lie algebra $\mathcal{G}$ with respect to the Lie product (3). Introducing the current operator Lie algebra $C^1_{\mathbf{S}}(\mathcal{G})$ and further its central extension by means of the standard Maurer – Cartan cocycle (7), one can consider integrable operator dynamical systems [8] of the form (22), based on solutions to the characteristic operator equation (11). As a result in such a way one can construct new integrable many dimensional nonlocal differential-difference dynamical systems. For instance, a suitable operatorial variant of Example is given by an element $l \in C^1_{\mathbf{S}}(\mathcal{G}_+)$, where

$$l := l[\hat{u}] = \mathrm{i}\alpha T + \mathrm{i}\varepsilon^{-1}\hat{u}\mathbf{1}. \tag{25}$$

Here $\hat{u} \in \hat{M} \subset C^{\infty}(\mathbf{S}^1;\mathcal{B})$ and

$$\mathcal{B} := \left\{ \sum_{i\in\mathbf{Z}}^{i<<\infty} u_i\xi^i \,:\, u_i \in \mathcal{S}(\mathbf{R};\bar{M}), i \in \mathbf{Z} \right\},$$

where $\bar{M} \subset C^{\infty}(\mathbf{S}^1;\mathbf{R})$ – some basic functional manifold, and by definition,

$$[\xi, a(y)] := \frac{\partial a(y)}{\partial y}, \quad \tau(a) := \int_{\mathbf{R}} dy \,\mathrm{res}_{\xi} a(y)$$

for any $a \in \mathcal{B}$, $y \in \mathbf{R}$. Having assumed now that $\hat{u} := u + \mathrm{i}\delta\xi \in \hat{M}$, from (22) and (11) one gets a new (different of that from [6]) integrable nonlocal differential-difference Kadomtsev – Petviashvili equation, having important applications in hydrodynamics and plasma physics. On the other hand, if the automorphism $T : \mathcal{A} \to \mathcal{A}$ is defined as

$$(Ta)(y) := (T_{\delta}a)(y) := a(y + \mathrm{i}\delta), \quad y \in \mathbf{R},$$

for any $a \in \mathcal{A}$, $\delta \in \mathbf{R}_+$, and the algebra $\mathcal{A}$ is $\mathbf{R} \ni x$ — independent, then the characteristic equation (11) will take such a simple form:

$$[\nabla\gamma(l), l] = 0. \tag{26}$$

The equation (26) evidently admits the following general solutions:

$$\gamma_n^{(p)} := \mathrm{Tr}\, l_{(p)}^{n/p} \in I(\mathcal{G}^*), \tag{27}$$

where $n \in \mathbf{Z}_+$, and by definition, for any $p \in \mathbf{Z}_+$

$$\mathrm{Tr}\, A_{(p)}(T) := \tau(a_0^{(p)}), \quad T \Rightarrow T_{\delta/p},$$

with

$$A_{(p)}(T) := \sum_{k\in\mathbf{Z}}^{k<<\infty} a_k^{(p)} T_{\delta/p}^k \in \mathcal{A}.$$

For instance, in (25) the element $l \in \mathcal{G}_+^*$ can be represented equivalently as

$$l = l_{(p)} = \mathrm{i}\alpha(T_{\delta/p})^p + \mathrm{i}\varepsilon^{-1}\hat{u}\mathbf{1}$$

for any $p \in \mathbf{Z}_+$, that entails existence of the nontrivial p-th root $l_{(p)}^{1/p} \in \mathcal{G}$ what makes it possible to compute Casimir functionals (27) as nontrivial ones. Having reduced now the element (25) upon the degenerate subspace

$$\hat{M}_0 = \{\hat{u} \in \hat{M} : u_j \neq 0, j \neq 0, 1, \ u_0 = u, \ u_1 = \mathrm{i}\varepsilon\xi\},$$

from (22) and (27) one can retrieve successively the nonlocal differential-difference Korteweg - de Vries hierarchy before constructed in [7]:

$$\frac{du}{dt_1} = -u_x, \quad \frac{du}{dt_2} = 2uu_x - Hu_{xx},$$

$$\frac{du}{dt_3} = \frac{1}{4}u_{xxx} + 3u^2u_x + \frac{3}{2}H(uu_x)_x + \frac{3}{2}u(Hu_x)_x + \frac{3}{4}H^2u_{xxx}, ...,$$

where $t_j \in \mathbf{R}, j \in \mathbf{Z}_+$, — the corresponding evolution parameters and $u \in \bar{M}$.

Note. It is evident from the consideration above that our construction contains as a partial case the standard local theory of integrable differential-difference equations [4] if one to put the parameter $x \in \mathbf{Z} \subset \mathbf{R}$ and take an associative algebra $\mathcal{A}$ equal to $C_{\mathbf{Z}}(\mathcal{B})$ with the automorphism $T : \mathcal{A} \to \mathcal{A}$, acting as

$$(Ta)_n := a_{n+1}, \ n \in \mathbf{Z},$$

for any $a \in \mathcal{A} := C_{\mathbf{Z}}(\mathcal{B})$. The corresponding linear mapping $\tau : \mathcal{A} \to \mathbf{C}$ is defined as follows:

$$\tau(a) := \sum_{n \in \mathbf{Z}} \int_{\mathbf{R}} dy \operatorname{res}_\xi a_n(y;\xi),$$

for any $a = \{a_n(y;\xi) \in \mathcal{B} : n \in \mathbf{Z}, y \in \mathbf{S}^1\} \in \mathcal{A}$. The corresponding operator Lie algebra $\mathcal{G}$ of homomorphisms of the space $\mathcal{A}$ can be naturally centrally extended by means of the standard Maurer – Cartan 2-cocycle (7) in the variable $y \in \mathbf{S}^1$. The resulting equation on Casimir functionals $\gamma \in I(\hat{\mathcal{G}}^*)$ is written now as

$$\frac{d\nabla\gamma(l)}{dy} = [l, \nabla\gamma(l)] \tag{28}$$

for any $l \in \mathcal{G}^*, y \in \mathbf{S}^1$, and Lax type equations (22) as

$$\frac{dl}{dt} = \left[l - \frac{d}{dy}, (\nabla\gamma(l))_+\right], \tag{29}$$

$t \in \mathbf{R}$ – an evolution parameter. For instance, the element

$$l := l[u, v] := T + v + T^{-1}u \in \mathcal{G}^*$$

generates via solution of the equation (28) and substituting a solution into (29) a generalized Toda chain. The latter in the case of independence on the parameter $y \in \mathbf{S}^1$ reduces into the standard Toda system on the manifold $M \subset C_{\mathbf{Z}}(\mathbf{R}^2)$:

$$\frac{du}{dt} = u(Tv - v), \quad \frac{dv}{dt} = v(T^{-1}u - u),$$

which is well known to be Lax type integrable [5, 9] bi-Hamiltonian flow on the discrete manifold M.

6. Local differential-diference equations: an alternative approach. Below we shall consider the Lie-algebraic aspects of differential-difference integrable flows associated with the following generalized linear matrix problem:

$$f_{n+1} = l_n[u;\lambda]f_n, \tag{30}$$

where $f \in l_\infty(\mathbf{Z};\mathbf{C}^p)$, $l_n := l_n[u;\lambda] \in G := GL_p(\mathbf{C})$ for all $n \in \mathbf{Z}_N$, with $N \in \mathbf{Z}_+$ being fixed, $\lambda \in \mathbf{C}$ — a "spectral" parameter and $u \in M \subset C_{\mathbf{Z}_N}(\mathbf{R}^m)$ — an apriori given discrete finite-dinemsional manifold. To describe local integrable differential-difference equations associated with the linear problem (30) it is necessary to study in detail the natural action of the product-group $G^N := \bigotimes_{i=1}^N G$ on the phase space $M_G = \{l_n \in G : n \in \mathbf{Z}_N\}$:

$$G^N \times M_G :\longrightarrow M_G,$$

given as follows:

$$\{g_n : n \in \overline{1,N}\} \times \{l_n : n \in \mathbf{Z}_N\} := \{g_{n+1}l_n g_n^{-1} : n \in \mathbf{Z}_N\}. \tag{31}$$

A functional $\gamma \in \mathcal{D}(M_G)$ is invariant with respect to the action (31) iff the discrete equation

$$\nabla\gamma(l_{n+1})l_{n+1} = l_n\nabla\gamma(l_n), \tag{32}$$

where $\nabla\gamma(l_n) \in \mathcal{G}^*$, $\mathcal{G}$ – the Lie algeabra of the group G, holds for all $n \in \mathbf{Z}_N$. From the relationship (32) one finds readily that the quantity $S_n := S_n(l) = \nabla\gamma(l_n)l_n$, $n \in \mathbf{Z}_N$, called the monodromy matrix, satisfies the following difference equation:

$$S_{n+1}(l)l_n = l_n S_n(l) \tag{33}$$

for $n \in \mathbf{Z}_N$ and any $l \in M_G$. Taking now the usual Sp-operation of the both sides of (33) one gets immediately that all functionals

$$\gamma_j(S_N) := \operatorname{Sp} S_n^j(l), \quad j \in \mathbf{Z}_+,$$

are independent of the discrete parameter $n \in \mathbf{Z}_N$ and are invariant evidently with respect to the standard Lie group G-action on the basic element $S_n \in \mathcal{G}^* \cdot G$ due to (33). It is an easy task to verify tha for all $j \in \mathbf{Z}_+$ the result

$$\nabla\gamma_j(S_N)(l_n) = S_n^j(l)l_n^{-1}$$

holds for any $n \in \mathbf{Z}_N$. Whence one can formulate the following important statement.

Theorem 2. *The hierarchy of invariant with respect to the group action (31) the functionals $\gamma_j \in \mathcal{D}(M_G)$, $j \in \mathbf{Z}_+$, is given exactly by the following expression:*

$$\gamma_j := \gamma_j(S_N) = \operatorname{Sp} S_N^j(l), \tag{34}$$

where due to (33) for $S_N(l)$ one can take such a matrix quantity:

$$S_N(l) = \prod_{n=1}^{N-1} l_n[u;\lambda],$$

being the real monodromy matrix for the linear problem (30).

It is necessary to point out that an assertion similar to Theorem 2 was also formulated without proof in [10], devoted to the $\mathcal{R}$-matrix aproach to differential-difference equations.

Now we shall be interested in a regular procedure of generating the integrable differential-difference flows associated naturally with the hierarchy of invariant functionals (34) on the discrete manifold M_G. Consider the associative algebra $\mathcal{A}_\lambda$ of all matrices $S_N(l) \in \mathcal{G}^* \cdot G$ which is converted into the Lie algebra $\mathcal{G}_\lambda$ with the standard Lie product (3) on which could be defined an $\mathcal{R}$-structure with the Lie product $[.,.]_{\mathcal{R}}$. Then evidently all functionals (34) are Casimir ones on the space $\mathcal{G}_\lambda^*$ generating on M_G the following Lax type Poisson flows:

$$\frac{dS_N}{d\tau_j} = [S_N, \mathcal{R}\nabla\gamma_j(S_N)] \tag{35}$$

for all $j \in \mathbf{Z}_+$. But these flows (35), as is easy to verify, generate the corresponding flows on element $l \in M_G$

$$\frac{dl_n}{d\tau_j} = \{\mathcal{R}(\Psi_{n+1}^{-1}(l)\nabla\gamma_j(S_N)\Psi_{n+1}(l))\}l_n -$$

$$- l_n\{\mathcal{R}(\Psi_n^{-1}(l)\nabla\gamma_j(S_N)\Psi_n(l)\},$$

with

$$\Psi_n(l) := \prod_{j=1}^{n} l_j[u;\lambda],\ n \in \mathbf{Z},$$

which do not conserve the invariant functionals $\gamma_k \in \mathcal{D}(M_G)$ for all $j, k \in \mathbf{Z}_+$. To remedy this problem let us generate hierarchy of flows on $\mathcal{G}_\lambda$ by means of making use of the well known second [9, 10] nonlinear in $S_N \in \mathcal{G}_\lambda^*$ $\mathcal{R}$-structure on the Lie algebra $\mathcal{G}_\lambda$:

$$\mathcal{R}(S_N)a := \mathcal{R}^*[S_N, a]S_N + [S_N, \mathcal{R}(aS_N)] \tag{36}$$

for any $a \in \mathcal{G}_\lambda$. As a result of using (36) the following new hierarchy of Lax type flows on M_G is easily obtained:

$$\frac{dS_N}{dt_j} = [S_N, \mathcal{R}(\nabla\gamma_j(S_N)S_N)] \tag{37}$$

for all $j \in \mathbf{Z}_+$. Flows (37) are evidently naturally associated with the next flows on M_G: for any $j \in \mathbf{Z}_+$

$$\frac{dl_n}{dt_j} = \mathcal{P}_{n+1}(l)l_n - l_n\mathcal{P}_n(l), \tag{38}$$

where for all $n \in \mathbf{Z}_N$ we defined

$$\mathcal{P}(l) := \mathcal{R}(\Psi_n^{-1}(l)\nabla\gamma_j(S_N)S_N\Psi_n(l)). \tag{39}$$

Since the following identity

$$\nabla\gamma_j(l_n) = \Psi_n^{-1}(l)\nabla\gamma_j(S_N)S_N\Psi_n(l)$$

holds for all $j \in \mathbf{Z}_+, n \in \mathbf{Z}$, the evolutions (38) can be rewritten down as flows

$$\frac{dl_n}{dt_j} = (\mathcal{R}\nabla\gamma_j(l_{n+1}))l_n - l_n(\mathcal{R}\nabla\gamma_j(l_n))$$

on M_G for all $j \in \mathbf{Z}_+, n \in \mathbf{Z}$. It is a simple task to verify that for all $k, j \in \mathbf{Z}_+$

$$\frac{d\gamma_k}{dt_j} = \left(\nabla\gamma_k(l_n), \frac{dl_n}{dt_j}\right) = (\nabla\gamma_k(l_n), (\mathcal{R}\nabla\gamma_j(l_{n+1}))l_n - l_n\nabla\gamma_j(l_n)) =$$

$$= (\mathcal{R}^*(l_n\nabla\gamma_k(l_n)), \nabla\gamma_j(l_{n+1}) - (\mathcal{R}^*(\nabla\gamma_k(l_n))l_n), \nabla\gamma_j(l_n)) =$$

$$= (\mathcal{R}^*(l_{n-1}\nabla\gamma_k(l_{n-1})), \nabla\gamma_j(l_n)) - (\mathcal{R}^*(\nabla\gamma_k(l_n)l_n), \nabla\gamma_j(l_n)) =$$

$$= (l_{n-1}\nabla\gamma_k(l_{n-1}) - \nabla\gamma_k(l_n)l_n), \mathcal{R}\nabla\gamma_j(l_n) \equiv 0,$$

since the invariant functionals $\gamma_k \in \mathcal{D}(M_G)$, $k \in \mathbf{Z}_+$, satisfy in virtue of (32) the condition $\nabla\gamma_k(l_n)l_n = l_{n-1}\nabla\gamma_k(l_{n-1})$ for all $n \in \mathbf{Z}$. Thus the above built evolution Lax type equations (38), (39) possess the hierarchy of invariant functionals (34) being in involution with respect to the correspondingly reduced on the finite-dimensional manifold M the Poisson bracket (37) on the adjoint space $\mathcal{G}^*_\lambda$. This ends our selfcontained analysis of the Lie-algebraic integrability of the local differential-difference Lax type flows on the group manifold M_G.

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